3 The Law of Large Numbers: Bernoulli

Texts

David, F.N.(1962) *Games, Gods, and Gambling* Dover, New York.
Hacking, Ian (1975) *The emergence of probability* Cambridge University Press, Cambridge.
The History of Statistics : The Measurement of Uncertainty Before 1900 Stephen M. Stigler
Hacking, Ian (1990) *The Taming of Chance* Cambridge University Press, Cambridge.
Bernoulli, Jacques (1713) *Ars Conjectandi* (See Hacking(1975) or Statistics Library.)
Keynes,J.M(1921) *A Treatise on Probability,* London

1 Ars Conjectandi

The title *Ars Conjectandi*, Art of Conjecturing, still ring a beautiful chime down the three centuries since James Bernoulli(1654-1705) master work was published posthumously in 1713. The book includes Huygens's treatise, many combinatoric methods, and applications to economics, morality, and politics. However, it is best known for Bernoulli's " golden theorem":

This is therefore the problem that I now want to publish here, having considered it closely for a period of twenty years, and it is a problem of which the novelty, as well as the high utility, together with its grave difficulty, exceed in weight and value all the remaining chapters of my doctrine.

The following quotes are via Stigler(1986):

To illustrate this by an example, I suppose that without your knowledge there are concealed in an urn 3000 white pebbles and 2000 black pebbles, and in trying to determine the numbers of these pebbles you take out one pebble after another (each time replacing the pebble you have drawn before choosing the next, in order not to decrease the number of pebbles in the urn), and that you observe how often a white and how often a black pebble is withdrawn. The question is, can you do this so often that it becomes ten times, one hundred times, one thousand times, etc., more probable (that is, it be morally certain) that the numbers of whites and blacks chosen are in the same 3 : 2 ratio as the pebbles in the urn, rather than in any other different ratio?
(Bernoulli, 1713, pp. 225- 226)

Bernoulli recognized that we could not count on determining the ratio exactly but would have to content ourselves with an approximation to the true ratio:

To avoid misunderstanding, we must note that the ratio between the number of cases, which we are trying to determine by experiment, should not be taken as precise and indivisible (for then just the contrary would happen, and it would become less probable that the true ratio would be found the more numerous were the observations). Rather, it is a ratio taken with some latitude, that is, included within two limits which can be made as narrow as one might wish. For instance, if in the example of the pebbles alluded to above we take two ratios 301/200 and 299/200 or 3001/2000 and 2999/2000, etc., of which one is immediately greater and the other immediately less than the ratio 3: 2, it will be shown that it can be made more probable, that the ratio found by often repeated experiments will fall within these limits of the 3 : 2 ratio rather than outside them. (Bernoulli, 1713, pp. 226-227)

Stigler discusses Bernoulli's mathematical argument. It depends on the fact that for a binomial random variable X with parameters n,p
$P\{X = k\}/P\{X = k-1\}$ is decreasing in *k* for $k > np$. In addition,
$P\{X = k\}/P\{X = k-1\} \to 0$ as $n \to \infty$ for $k > n(p+\varepsilon)$.
The first fact shows that the tail contribution $P\{X \ge k\}$ is about the same size as $P\{X = k\}$, and the second fact shows that $P\{X \ge n(p+\varepsilon\}$ is negligible compared to $P\{X = [np]\}$, and this is enough to prove the Law of Large Numbers,
$P\{|X - np| \ge n\varepsilon\} \to 0$ as $n \to \infty$.

Actually, the mathematics is less interesting than the philosophical consequences, and Bernoulli was well aware of these; from Stigler:

Bernoulli began the discussion leading up to his theorem by noting that, in games employing homogeneous dice with similar faces or urns with equally accessible tickets of different colors, the a priori determination of chances was straightforward. One would simply enumerate the possible cases and take the ratio of the number of "fertile" cases to the total number of cases, whether "fertile" or "sterile." But, Bernoulli asked, what about problems such as those involving disease, weather, or games of skill, where the causes are hidden and the enumeration of equally likely cases impossible? In such situations, Bernoulli wrote, "It would be a sign of insanity to attempt to learn anything in this manner. '

Instead, Bernoulli proposed to determine the probability of a fertile case a posteriori: "For it should be presumed that a particular thing will occur or not occur in the future as many times as it has been observed, in similar circumstances, to have occurred or not occurred in the past" (1713, p.224). The proportion of favorable or fertile cases could thus be determined empirically. Now this empirical approach to the determination of chances was not new with Bernoulli, nor did he consider it to be new. What was new was Bernoulli's attempt to give formal treatment to the vague notion that the greater the accumulation of evidence about the unknown proportion of cases, the closer we are to certain knowledge about that proportion.

2 Leibnitz

Leibnitz, in a letter to Bernoulli in 1703, made the following objection to Bernoulli's new method of determining probabilities from empirical frequencies.
From the anti-frequentist Keynes(1921):

Leibniz's reply goes to the root of the difficulty.
"The calculation of probabilities is of the utmost value, he says, but in statistical inquiries there is need not so much of mathematical subtlety as of a precise statement of all the circumstances. The possible contingencies are too numerous to be covered by a finite number of experiments, and exact calculation is, therefore, out of the question. Although nature has her habits, due to the recurrence of causes, they are general, not invariable. Yet empirical calculation, although it is inexact, may be adequate in affairs of practice."

Leibniz's actual expressions (in a letter to Bernoulli, December 3, 1703) are as follows:

Utilissima cst aestimatio probabilitatum, quanquam in exemplis juridicis politicisque plerumque non tam subtili calculo opus est, quam accurate omnium circumstantiarum enumeratione. Cum empirice aestimamus probabilitates per experimenta successuum, quaeris an ea via tandem aestimatio perfecte obtineri possit. Idque a Te repertum scribis. Difficultas in eo mihi inesse videtur, quod contingentia seu quae infinitis pendent circumstantiis, per finita experimenta determinari non poseunt ; natura quidem suas habet consuetudines, natas ex reditu causarum, sed non nisi (some greek). Novi morbi inundant subinde humanum genus, quodsi ergo de mortibus quotcunque experimenta feceris, non ideo naturae rerum limites posuisti, ut pro futuro variarenon possit. Etsi autem empirice non posset haberi perfecta aestimatio, non ideo minus empirica aestimatio in praxi utilis et sufficiens foret.